MATH 3060 Assignment 3 solution

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1. (a) It is clear that $d(x,y) = d(y,x)$, $d(x,y) \geq 0$ and $d(x,y) = 0$ if and only if $x = y$. Let $x, y, z \in R_+$, then

$$\begin{aligned} d(x,z) &= \left|\frac{1}{x} - \frac{1}{z}\right| \\ &= \left|(\frac{1}{x} - \frac{1}{y}) + (\frac{1}{y} - \frac{1}{z})\right| \\ &\leq \left|\frac{1}{x} - \frac{1}{y}\right| + \left|\frac{1}{y} - \frac{1}{z}\right| \\ &= d(x,y) + d(y,z). \end{aligned}$$

   (b) It is clear that $d_1(x,y) = d_1(y,x)$, $d_1(x,y) \geq 0$ and $d_1(x,y) = 0$ if and only if $x = y$. Next we suppose $x, y, z \in X$, note that

$$d_1 = \frac{d}{1+d} = 1 - \frac{1}{d}.$$

   We have

$$\begin{aligned} & d_1(x,y) + d_1(y,z) \\ =& 2 - \frac{1}{1+d(x,y)} - \frac{1}{1+d(y,z)} \\ \geq& 2 - \frac{1}{1+d(x,y)} - \frac{1}{1+d(x,y)+d(y,z)} \\ \geq& \left(1 - \frac{1}{1+d(x,y)}\right) - \left(1 - \frac{1}{1+d(x,y)+d(y,z)}\right) \\ \geq& 0 + d_1(x,z) \\ =& d_1(x,z). \end{aligned}$$

2. (a) No, condisder the function (which reduced to x^n if $a = 0, b = 1$)

$$f_n(x) = \left(\frac{x-a}{b-a}\right)^n$$

Then $d_1(f_n, 0) = (b-a)(n+1)^{-1}$, $d_2(f_n, 0) = (b-a)^{1/2}(2n+1)^{-1/2}$, and

$$\frac{d_2(f_n, 0)}{d_1(f_n, 0)} = O(n^{1/2})$$

is unbounded.

(b) Yes, beacause by Hölder's inequality

$$\begin{aligned} &d_1(f, g) \\ =&\int_a^b |f-g||1| \\ \leq&\left(\int_a^b |f-g|^2\right)^{1/2} \left(\int_a^b 1\right)^{1/2} \\ =&(b-a)^{1/2} d_2(f, g). \end{aligned}$$

3. It is clear that $d(f, g) = d(g, f)$, $d(f, g) \geq 0$ and $d(g, f) = 0$ if and only if $f = g$. Moreover, for $f, g, h \in C^1[a, b]$ and $x, y \in [a, b]$

$$\begin{aligned} &d(f, g) + d(g, h) \\ =&|f-g|_\infty + |f-g|_\infty + |f'-g'|_\infty + |g-h|_\infty + |g'-h'| \\ \geq&|f(x)-g(x)| + |f'(y)-g'(y)| + |g(x)-h(x)| + |g'(y)-h'(y)| \\ \geq&|f(x)-h(x)| + |f'(y)+h'(y)|, \end{aligned}$$

since x, y are arbitrary, we see that $d(f, g) + d(g, h) \geq d(f, h)$.

Next, for

$$\begin{aligned} f_k(x) &= \int_0^{1/k} \sin(ktx)dt \\ &= \frac{1}{k}\int_0^1 \sin(tx)dt, \end{aligned}$$

we have

$$f_k'(x) = \frac{1}{k}\int_0^1 t\cos(tx)dt.$$

We thus see that $|f_k|_\infty, |f_k'|_\infty < 1/k$, and so

$$d(f_k, 0) < \frac{2}{k},$$

thus f_k converges to the zero function.

4. (a) It is clear that $d_\infty(f,g) = d_\infty(g,f)$ and $d_\infty(f,g) \geq 0$. If $d_\infty(f,g) = 0$, then $0 \sup|f-g| = 0$, which means $f = g$. Moreover, suppse $f, g, h \in C[a,b]$ and $x \in [a,b]$

$$\begin{aligned}
&d_\infty(f,g) + d_\infty(g,h) \\
=&\sup|f-g| + \sup|g-h| \\
\geq&|f(x)-g(x)| + |g(x)-h(x)| \\
\geq&|f(x)-h(x)|.
\end{aligned}$$

Since x is arbitrary, we have $d_\infty(f,g) + d_\infty(g,h) \geq d_\infty(f,h)$.

(b) Let $\epsilon > 0$, and take $0 < \delta < \epsilon/(b-a)$. If $f, g \in C^1[a,b]$ and $d_\infty(f,g) < \delta$, then

$$\begin{aligned}
d_\infty(Sf, Sg) &= \sup_x \int_a^x f(t) - g(t) dt \\
&\leq \sup_x \int_a^x |f(t) - g(t)| dt \\
&\leq \sup_x \int_a^x \delta dt \\
&= \delta(x-a) \\
&\leq \delta(b-a) \\
&< \epsilon.
\end{aligned}$$

Therefore, S is (uniformly) continuous on $C^1[a,b]$.